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Concrete Hilbert Spaces for Quantum Systems with Infinitely Many Degrees of Freedom

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We demonstrate a method to describe quantum systems with infinitely many degrees of freedom in concrete Hilbert spaces, using the electromagnetic radiation field as a well-known example of such a system. Since our method is not only applicable to the case of countably many but even to the case of uncountably many degrees of freedom, there is no need for a finite quantization volume in radiation theory.

1. INTRODUCTION

The quantum theory of radiation was developed by Dirac (1926, 1927b) using the transformation theory as a mathematical basis (Dirac, 1927a). A description by means of a theory of separable Hilbert spaces was worked out by von Neumann (1932/1981, pp. 135 ff.). In this paper we use von Neumann's mathematical concept generalized in the following way (see, e.g., Gerlich, 1977, 1987, 1992): Instead of the Lebesgue measure, we use probability measures and corresponding integration spaces in order to build infinite-dimensional products. To represent uncountably many degrees of freedom, we dispense with the separability of Hilbert spaces in quantum theory.

2. THE CONVENTIONAL CONCEPT TO QUANTIZE THE RADIATION FIELD

As a solution of the Maxwell equations in source-free space the electric and magnetic fields are derived from the potentials **A** and Φ via the relations

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$\mathbf{E} = -\text{grad}\,\Phi - \partial/\partial t\,\mathbf{A}$ and $\mathbf{B} = \text{rot}\,\mathbf{A}$. The radiation gauge div $\mathbf{A} = \Phi = 0$ leads to a homogeneous wave equation for the vector potential

$$\Delta \mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = 0 \tag{1}$$

Assuming that the discussed field is confined with a spatial volume of finite size L^3, we can expand $\mathbf{A}$ in a Fourier series

$$\mathbf{A}(\mathbf{r}, t) = \sum_{k=1}^{\infty} q_k(t)\overline{\mathbf{A}}_k(\mathbf{r}) \tag{2}$$

with an appropriate set of vector mode functions $\overline{\mathbf{A}}_k(\mathbf{r})$. Each Fourier amplitude q_k obeys the dynamical law of a harmonic oscillator of unit mass and frequency ω_k,

$$\ddot{q}_k = -\omega_k^2 q_k \tag{3}$$

The field's energy

$$\frac{1}{2}\epsilon_0 \int_{L^3} (\mathbf{E}^2(\mathbf{r}, t) + c^2\mathbf{B}^2(\mathbf{r}, t))\, d\mathbf{r}$$

$$= \frac{1}{2}\sum_{k=1}^{\infty} (p_k^2 + \omega_k^2 q_k^2), \qquad p_k := \dot{q}_k \tag{4}$$

is formally identical with the Hamilton function of a set of independent harmonic oscillators with unit mass, generalized coordinates $q_1, q_2, \ldots$, and frequencies $\omega_1, \omega_2, \ldots$. With the classical mechanical interpretation one can pass to quantum mechanics: Each oscillator q_k is described by a Hamiltonian

$$H_k = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial q_k^2} + \frac{\omega_k^2}{2} q_k^2 \tag{5}$$

which is defined on an appropriate subset of the integration space $L_k^2(\mu_L, \mathbf{R})$. Here $\mathbf{R}$ denotes the field of real numbers, μ_L the Lebesgue measure. The eigenfunctions $\psi_{M_k}^k$, $M_k \in \mathbf{N}_0$, of H_k form a complete orthonormal set in $L_k^2(\mu_L, \mathbf{R})$, where $\mathbf{N}_0$ denotes the set of natural numbers including 0. Hence the eigenfunctions $\psi_{M_1}^1 \cdots \psi_{M_n}^n$ of the n-particle Hamiltonian $\Sigma_{k=1}^n H_k$ are a complete orthonormal set in $L^2(\otimes^n \mu_L, \times^n \mathbf{R})$, which is the corresponding n-particle Hilbert space.

Since no infinite-dimensional Lebesgue measure exists, the limit $n \to \infty$ cannot be performed in the case of a representation with Lebesgue measure integration spaces (see, e.g., Gross, 1964, p. 52, or Skorohod, 1974, p. 102). Besides, the values $\Pi\, \psi_{M_k}^k(q_k)$ of the infinite products of energy eigenfunctions would not be finite in general. Instead of representing the whole system of oscillators in an appropriate "natural" state space of amplitudes $q_1, q_2, \ldots,$ one usually transforms to the space of occupation numbers $M_1, M_2, \ldots$. Mathematically, the latter is represented by the classical Hilbert space l^2 of sequences, which can be read as a space of complex functions in the measure-theoretic sense. Defined on $\mathbf{N}_0$, these functions are square integrable with respect to the counting measure. Here the limit $n \to \infty$ can be performed without any problem (see, e.g., von Neumann, 1932/1981, p. 143). In the sequel we are going to construct a state space of amplitudes for the complete system of oscillators. This can be done by substituting probability measures for the Lebesgue measure in the state spaces $L_k^2(\mu_L, \mathbf{R})$ of each oscillator.

3. HARMONIC OSCILLATORS IN PROBABILITY MEASURE INTEGRATION SPACES

To secure the convergence of infinite products, it is sufficient to have at most a finite number of factors different from one. The number of oscillators which are in an excited state that differs from the ground state ψ_0^k is always finite. Therefore we are looking for a class of unitary transformations U^{k*} that transform the spaces $L_k^2(\mu_L, \mathbf{R})$ into appropriate new spaces $L^2(w_k, \mathbf{R})$, where the relation

$$\phi_0^k := U^{k*}\psi_0^k \equiv 1 \tag{6}$$

must hold. To find an explicit form for the U^{k*}, we consider the eigenfunctions of the kth oscillator

$$\psi_{M_k}^k(q_k) = \frac{1}{(2^{M_k}M_k!)^{1/2}}\left(\frac{\omega_k}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{\omega_k}{2\hbar}q_k^2\right)H_{M_k}\left(\left(\frac{\omega_k}{\hbar}\right)^{1/2}q_k\right) \tag{7}$$

where

$$H_{M_k}(x_k) = (-1)^{M_k}e^{x_k^2}\frac{d^{M_k}}{dx_k^{M_k}}(e^{-x_k^2}) \tag{8}$$

are the Hermite polynomials. The corresponding ground state is

$$\psi_0^k(q_k) = \left(\frac{\omega_k}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{\omega_k}{2\hbar}q_k^2\right) \tag{9}$$

Thus we define

$$U^k\phi_{M_k}^k|_{q_k} = [\rho_k(q_k)]^{1/2}\phi_{M_k}^k(q_k) = \psi_{M_k}^k(q_k) \tag{10}$$

$$\rho_k(q_k) = \left[\left(\frac{\omega_k}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{\omega_k}{2\hbar}q_k^2\right)\right]^2 \tag{11}$$

$$\phi^k_{M_k}(q_k) = \frac{1}{(2^{M_k} M_k!)^{1/2}} H_{M_k}\left(\left(\frac{\omega_k}{\hbar}\right)^{1/2} q_k\right) \tag{12}$$

The functions $\{\phi^k_{M_k} | M_k \in \mathbf{N}_0\}$ are a complete orthonormal set in $L^2(w_k, \mathbf{R})$.

Since ρ_k: $\mathbf{R} \to \mathbf{R}$ is a nonnegative Lebesgue measurable function, the relation $w_k := \rho_k \mu_L$ defines a new measure on the σ-algebra $\mathcal{A}_L$ of Lebesgue measurable subsets of $\mathbf{R}$ (see, e.g., Rudin, 1966/1987, p. 23). For arbitrary sets $A \subset \mathcal{A}_L$ we have

$$w_k(A) = \int_A w_k(dq_k) = \int_A \rho_k(q_k)\, \mu_L(dq_k) \tag{13}$$

The transformation U^{k*} changes the measure space $(\mathbf{R}, \mathcal{A}_L, \mu_L)_k$ into $(\mathbf{R}, \mathcal{A}_L, w_k)$. Here w_k is a probability measure

$$w_k(\mathbf{R}) = \int_{-\infty}^{\infty} w_k(dq_k) = \int_{-\infty}^{\infty} \left(\frac{\omega_k}{\pi\hbar}\right)^{1/2} \exp\left(-\frac{\omega_k}{\hbar} q_k^2\right) \mu_L(dq_k) = 1 \tag{14}$$

The latter property allows the construction of an infinite-dimensional product measure and thus of an appropriate state space of amplitudes for the whole number of harmonic oscillators.

Given the maps U^k explicitly, one can calculate the transformed operators $\tilde{A}_k$ from the original ones A_k,

$$\begin{array}{ccc} L^2_k(\mu_L, \mathbf{R}) & \xrightarrow{\;U^{k*}\;} & L^2(w_k, \mathbf{R}) \\ A_k \downarrow & & \downarrow U^{k*} \circ A_k \circ U^k =: \tilde{A}_k \\ L^2_k(\mu_L, \mathbf{R}) & \xrightarrow[U^{k*}]{} & L^2(w_k, \mathbf{R}) \end{array}$$

We get

$$\widetilde{q_k} = q_k, \qquad \widetilde{p_k} = i\omega_k q_k + p_k \tag{15}$$

the amplitude operators

$$\widetilde{a_k} = \left(\frac{\hbar}{2\omega_k}\right)^{1/2} \frac{\partial}{\partial q_k}, \qquad \widetilde{a_k}^+ = \left(\frac{2\omega_k}{\hbar}\right)^{1/2} q_k - \left(\frac{\hbar}{2\omega_k}\right)^{1/2} \frac{\partial}{\partial q_k} \tag{16}$$

and, neglecting the zero-point energy, the new Hamiltonian

$$\widetilde{H}_k = \hbar\omega_k \widetilde{a}_k^{\,+} \widetilde{a}_k = \hbar\omega_k q_k \frac{\partial}{\partial q_k} - \frac{\hbar^2}{2} \frac{\partial^2}{\partial q_k^2} \tag{17}$$

The given transformed operators $\tilde{A}_k$ are defined on the linear span of the complete orthonormal set $\{\phi^k_{M_k} | M_k \in \mathbf{N}_0\}$, which clearly is a dense subset of $L^2(w_k, \mathbf{R})$. The commutation relations remain unchanged, because the maps U^k are unitary, i.e., linear with domain $L^2_k(\mu_L, \mathbf{R})$. For the same reasons Hermitian operators in $L^2_k(\mu_L, \mathbf{R})$ transform to Hermitian operators in $L^2(w_k, \mathbf{R})$. As we are going to prove in the next section, the operators $\widetilde{q}_k$, $\widetilde{p}_k$, and $\widetilde{H}_k$ are even self-adjoint.

4. INFINITELY MANY OSCILLATORS IN A STATE SPACE OF AMPLITUDES

With the new measure spaces $(\mathbf{R}, \mathcal{A}_L, w_k)$ of all amplitudes q_k one can construct the product measure space $(\times_{k\in\mathbf{N}}\mathbf{R}, \otimes_{k\in\mathbf{N}}\mathcal{A}_L, \otimes_{k\in\mathbf{N}}w_k) = \otimes_{k\in\mathbf{N}}(\mathbf{R}, \mathcal{A}_L, w_k)$ (see, e.g., Bauer, 1968/1974). The σ-algebra $\otimes_{k\in\mathbf{N}}\mathcal{A}_L$ is generated by the cylinder sets with measurable basis which are subsets of $\times_{k\in\mathbf{N}}R$, i.e., by the class of subsets $\times_{k\in\mathbf{N}}A_k$ of the Cartesian product $\times_{k\in\mathbf{N}}\mathbf{R}$, $A_k \in \mathcal{A}_L$, where only a finite number of sets A_k differ from $\mathbf{R}$. The corresponding Hilbert space of square-integrable functions $L^2(\otimes_{k\in\mathbf{N}}w_k, \times_{k\in\mathbf{N}}\mathbf{R}) =: L^2(w)$ is an appropriate amplitude state space for the entire set of oscillators. Each operator on $L^2(w_k, \mathbf{R})$ can be interpreted as an operator on $L^2(w)$. Hence, $\Sigma_{k=1}^{\infty} \widetilde{H}_k$ is the Hamiltonian of the entire system. In consideration of the finite energy of the entire system, $\Sigma_{k=1}^{\infty} M_k < \infty$, the finite products $\Pi_{k\in\mathbf{N}} \phi^k_{M_k}$ of energy eigenfunctions converge, because only a finite number of factors, say p, is not equal to one; we have

$$\sum_{k=1}^{\infty} \widetilde{H}_k \prod_{i\in\mathbf{N}} \phi^i_{M_i} = \sum_{k=1}^{\infty} M_k\hbar\omega_k \prod_{i\in\mathbf{N}} \phi^i_{M_i}$$

$$= \sum_{l=1}^{p} \hbar\omega_{l_p} M_{l_p} \phi^{l_1}_{M_{l_1}} \cdots \phi^{l_p}_{M_{l_p}}, \qquad p \in \mathbf{N} \tag{18}$$

The linear span of the complete orthonormal set of energy eigenfunctions is dense in $L^2(w)$ and part of the domain of all interesting operators. The Hermitian character of the multiplication operator $\widetilde{q}_k$ is evident, whereas the examination of $\widetilde{p}_k$ and $\widetilde{H}_k$ is less trivial. In the finite-dimensional case we made use of the fact that the maps U^k are unitary. In the infinite-dimensional case there is no Lebesgue-measure state space corresponding to $L^2(w)$. Therefore we have to look for another way of reasoning. Instead, we now make use of properties of the product measure $\otimes_{k\in\mathbf{N}} w_k$ and of their consequences for integration, which were studied in detail by Gerlich and Wulbrand (1978).

For continuously differentiable functions $\rho_i(q_i) \neq 0$ we have

$$\int\limits_{a_i \le q_i \le b_i} \frac{1}{\rho_k(q_k)} \frac{\partial}{\partial q_k} f(\ldots, q_k, \ldots) \bigotimes_{i\in\mathbf{N}} w_i(dq_i)$$

$$= \int\limits_{\substack{a_i \le q_i \le b_i \\ i\neq k}} f(\ldots, b_k, \ldots) \bigotimes_{\substack{i\in\mathbf{N} \\ i\neq k}} w_i(dq_i)$$

$$- \int\limits_{\substack{a_i \le q_i \le b_i \\ i\neq k}} f(\ldots, a_k, \ldots) \bigotimes_{\substack{i\in\mathbf{N} \\ i\neq k}} w_i(dq_i) \tag{19}$$

For functions $\psi, \phi \in L^2(w)$ that are partially differentiable with respect to q_k with derivative in $L^2(w)$, we obtain

$$\int \frac{\partial\overline{\psi}(q)}{\partial q_k} \phi(q)\, w(dq)$$

$$= \int \frac{1}{\rho_k(q_k)} \frac{\partial}{\partial q_k} (\rho_k(q_k)\overline{\psi}(q)\phi(q))\, w(dq)$$

$$- \int \overline{\psi}(q) \frac{1}{\rho_k(q_k)} \frac{\partial}{\partial q_k} (\rho_k(q_k)\phi(q))\, w(dq) \tag{20}$$

where we used the abbreviations $(\ldots, q_k, \ldots) =: q$ and $\otimes_{i\in\mathbf{N}} w_i =: w$. After integration by parts the first integral on the right side vanishes in the case of the Gaussian density (11). Finally, we find the formula

$$\int \frac{\partial\overline{\psi}(q)}{\partial q_k} \phi(q)\, w(dq) = \int \overline{\psi}(q) \left[\frac{-\partial\phi(q)}{\partial q_k} + 2\frac{\omega_k}{\hbar} q_k\phi(q) \right] w(dq) \tag{21}$$

which helps us to prove that $\widetilde{p_k}$ and $\widetilde{H_k}$ are Hermitian operators when defined on $L^2(w)$.

In order to prove that $\widetilde{q_k}$ is self-adjoint, we show that the domain of $\widetilde{q_k^*}$ is part of the domain of $\widetilde{q_k}$, i.e., $D(\widetilde{q_k^*}) \subset D(\widetilde{q_k})$. Choose an arbitrary element ϕ of $D(\widetilde{q_k^*})$. We then have $\phi^* = \widetilde{q_k^*}\phi \in L^2(w)$ and for all $\chi \in D(\widetilde{q_k})$

$$\int q_k\chi(q)\overline{\phi(q)}\, w(dq) = \int \chi(q)\overline{\phi^*(q)}\, w(dq)$$

$$\Leftrightarrow \int \chi(q)[\overline{q_k\phi(q) - \phi^*(q)}]\, w(dq) = 0 \tag{22}$$

Now choose

$$\chi(q) = \begin{cases} q_k\phi(q) - \phi^*(q) & \text{for} \quad q_k \in (a_k, b_k) \subset \mathbf{R}, (a_k, b_k) \text{ arbitrary} \\ 0 & \text{for} \quad q_k \notin (a_k, b_k) \end{cases} \tag{23}$$

Then $\chi \in D(\widetilde{q_k})$ and

$$\int_{\mathbf{R}^{N-\{k\}}} \int_{a_k}^{b_k} |q_k\phi(q) - \phi^*(q)|^2 \, w(dq) = 0 \tag{24}$$

It follows that $q_k\phi = \phi^*$ almost everywhere on $\mathbf{R}^{N-\{k\}} \times (a_k, b_k)$. Since (a_k, b_k) was chosen arbitrary, the latter relation even holds on $\mathbf{R}^N$. Using $\phi^* \in L^2(w)$, we get $\phi \in D(\widetilde{q_k})$ and $\widetilde{q_k^*}\phi = q_k\phi = \widetilde{q_k}\phi$; thus the proof is complete.

We now show that the momentum operator $\widetilde{p_k} = -i\hbar\, \partial/\partial q_k + i\omega_k q_k$, defined on

$$D(\widetilde{p_k}) = \begin{array}{l} \{\phi \in L^2(w) \,|\, \phi \in C_k^1, \text{ i.e., } \phi \text{ continuously} \\ \text{partially differentiable with respect} \\ \text{to } q_k;\ \partial/\partial q_k \phi \text{ and } q_k\phi \in L^2(w)\} \end{array} \tag{25}$$

is essentially self-adjoint. Since $\widetilde{p_k}$ is Hermitian, it is sufficient to show that $(\widetilde{p_k} \pm i1)(D(\widetilde{p_k}))$ is dense in $L^2(w)$ (see, e.g., Hellwig, 1964, pp. 153 ff.). To do this we consider the set

$$X = \begin{array}{l} \{\psi \in L^2(w) \,|\, \psi \in C_k^1,\ \psi \equiv 0 \text{ for those } q_k \\ \text{which are not an element of} \\ \text{a given individual interval } [a_k, b_k] \subset \mathbf{R}\} \end{array} \tag{26}$$

which is dense in $L^2(w)$. For each arbitrary chosen function $\psi \in X$ there exists a function $\phi \in D(\widetilde{p_k})$ that obeys the relation $(\widetilde{p_k} + i1)\phi = \psi$, i.e., $(\widetilde{p_k} + i1)(D(\widetilde{p_k})) \supset X$. Rescaling the q_k axis, we obtain the differential equation

$$\left(-i\frac{\partial}{\partial q_k} + i\hbar\omega_k q_k + i\right)\phi|_{(\ldots, q_k, \ldots)} = \psi(\ldots, q_k, \ldots) \tag{27}$$

with

$$\begin{aligned} \phi(\ldots, q_k, \ldots) &= -i\exp\left(q_k + \frac{\hbar\omega_k}{2} q_k^2\right) \\ &\quad \times \int_{q_k}^{\infty} \psi(\ldots, t_k, \ldots) \exp\left(-t_k - \frac{\hbar\omega_k}{2} t_k^2\right) w_k(dt_k) \end{aligned} \tag{28}$$

being a possible solution. Since ψ vanishes for all q_k not in $[a_k, b_k]$, so does

ϕ. Hence the chosen ϕ is an element of $L^2(w)$. With $(\widetilde{p}_k + i1)\phi = \psi \in L^2(w)$ we have $\widetilde{p}_k\phi \in L^2(w)$. Summarizing, we can state $X \subset (\widetilde{p}_k + i1)(D(\widetilde{p}_k))$, which means $(\widetilde{p}_k + i1)(D(\widetilde{p}_k))$ is dense in $L^2(w)$. In the same way we can prove that $(\widetilde{p}_k - i1)(D(\widetilde{p}_k))$ is dense in $L^2(w)$. The closures of the essentially self-adjoint operators $\widetilde{p}_k$ are uniquely defined and self-adjoint.

Until now none of our considerations has identified the system of infinitely many harmonic oscillators as a quantized radiation field. Specifying the interaction operator with another quantum system, for example, an atomic system with state space L^2_{atom}, we have that the system of oscillators becomes a quantized radiation field. The above representation allows us to describe the interaction in the state space $L^2(w) \otimes L^2_{\text{atom}}$ of the entire system in complete agreement with conventional considerations (see, e.g., von Neumann, 1932/1981, pp. 135ff.). In particular, we get the same probabilities for transitions between two different atomic energy eigenstates.

5. THE CONTINUUM OF MODES

Without the artificial boundary condition that the field is confined within a spatial volume of finite size we have to take a continuum of modes into account and thus a continuum of dynamical variables. As we will see in the sequel, the transition from countably infinitely to uncountably infinitely many coordinates is much less difficult than the transition from a finite to an infinite number, because the results of the discrete case can be transformed to the continuous one without any problem. Substituting $\mathbf{R}$ for the set of indices $\mathbf{N}$ when constructing the measure and Hilbert space, we obtain the state space $L^2(\otimes_{i\in\mathbf{R}} w_i, \times_{i\in\mathbf{R}} \mathbf{R})$ of uncountably many amplitudes $\{q_i \equiv q(i) | i \in \mathbf{R}\}$. As this space is not separable, there is no (Hilbert space) isomorphism of $L^2(\otimes_{i\in\mathbf{R}} w_i, \times_{i\in\mathbf{R}} \mathbf{R})$ onto l^2. For the new Hamiltonian we choose the enumerable sum of Hamiltonians of the one-particle Hilbert spaces $\Sigma_{i\in\mathbf{R}} \widetilde{H}_i$ [a definition of enumerable infinite sums and products was given by von Neumann (1938)]. Its eigenfunctions form a complete normalized orthogonal set $\{\Pi_{i\in\mathbf{R}} \phi^i_{M_i} | M_i \in \mathbf{N}_0, \Sigma_{i\in\mathbf{R}} M_i < \infty\}$ in $L^2(\otimes_{i\in\mathbf{R}} w_i, \times_{i\in\mathbf{R}} \mathbf{R})$. Notice that this Hamiltonian as an enumerable sum is not analogous to the energy of the entire system of uncountably many classical oscillators according to (4), which is

$$\frac{1}{2}\int (\widetilde{p}_i^{\,2} + \omega_i^2\widetilde{q}_i^{\,2})\, \mu_L(di) \tag{29}$$

Since equations (19)–(21) still hold in the enumerable case, all steps considering the Hermitian, respectively the self-adjoint, nature of certain operators can be transformed. Likewise the convergence of infinite sums and products can be transferred.

6. THE GENERAL CONCEPT AND OTHER APPLICATIONS

The above-described method of representing quantum systems with an infinite number of degrees of freedom in concrete Hilbert spaces can be used as a universal concept in all those cases where the following conditions hold: The one-particle states $\psi_n^k \in L_k^2(\mu_L, X)$ can be written as products of nonnegative, Lebesgue measurable functions $\sqrt{\rho_k}$ and functions $\phi_n^k \in L^2(w_k, X)$, the measures $w_k := \rho_k\mu_L$ are probability measures, and the new ground states ϕ_0^k equal unity. The density ρ_k then is given by

$$\sqrt{\rho_k(x)} := \psi_0^k(x)$$

Besides the one-dimensional harmonic oscillator, there are other simple but important quantum systems of this kind: Neglecting the movement of its center of mass, the normalized eigenfunctions of the hydrogen atom in relative coordinates are products of radial wave functions R_{nl} and the spherical harmonics Y_{lm} (see, e.g., Hittmair, 1972, pp. 88ff.)

$$\psi_{nlm}(r, \theta, \varphi) = R_{nl}(r)Y_{lm}(\theta, \varphi)$$

where $n = 1, 2, \ldots, l = 0, 1, \ldots, n, m = -l, \ldots, l$. Corresponding eigenvalues are the discrete energies

$$E_n = -\frac{\hbar^2}{2ma^2}\frac{1}{n^2}$$

a being the Bohr radius. If we set

$$\rho(r, \theta, \varphi) = (4a^{-3}e^{-2r/a})\left(\frac{1}{4\pi}\right) = \rho^R(r)\rho^Y(\theta, \varphi)$$

and

$$\phi_{nlm}(r, \theta, \varphi) = \left[\frac{1}{\rho(r, \theta, \varphi)}\right]^{1/2} \psi_{nlm}(r, \theta, \varphi)$$

$w = \rho(r, \theta, \varphi)r^2 \sin\theta \, dr \, d\theta \, d\varphi$ is a probability measure on $(\mathcal{A}_L, [0, \infty] \times [0, \pi] \times [0, 2\pi])$

$$w(\mathbf{R}) = \frac{4}{a^3}\int_0^\infty r^2 e^{-2r/a} dr \, \frac{1}{2}\int_0^\pi \sin\theta \, d\theta \, \frac{1}{2\pi}\int_0^{2\pi} d\varphi = 1$$

The ϕ_{nlm} are elements of $L^2(w, [0, \infty] \times [0, \pi] \times [0, 2\pi])$. Notice that the radial wave functions R_{nl} as well as the spherical harmonics Y_{lm} possess "natural densities" ρ^R, resp. ρ^Y, of probability measures w^R, resp. w^Y, with respect to the Lebesgue measure, i.e., $w = w^R \otimes w^Y$.

In the case of a one-dimensional infinite square-well potential (see, e.g., Hittmair, 1972, pp. 49ff.)

$$V(x) = \begin{cases} 0 & \text{for} \quad -a \leq x \leq a \\ V_0 \to +\infty & \text{for} \quad x < -a, \text{ resp. } x > a \end{cases}$$

the bound solutions of the stationary Schrödinger equation in the state space $L^2(\mu_L, \mathbf{R})$ are

$$\psi_n(x) = \left\{ \begin{matrix} 0 & \text{for} \quad x \leq -a \\ \left(\frac{1}{a}\right)^{1/2} \cos\left(\frac{(n+1)\pi x}{2a}\right) & \text{for} \quad -a < x < a \\ 0 & \text{for} \quad x \geq -a \end{matrix} \right\}, \quad n = 0, 2, 4, \ldots$$

$$\psi_n(x) = \left\{ \begin{matrix} 0 & \text{for} \quad x \leq -a \\ \left(\frac{1}{a}\right)^{1/2} \sin\left(\frac{(n+1)\pi x}{2a}\right) & \text{for} \quad -a < x < a \\ 0 & \text{for} \quad x \geq -a \end{matrix} \right\}, \quad n = 1, 3, 5, \ldots$$

with corresponding energy eigenvalues

$$E_n = \frac{\pi^2 \hbar^2}{8ma^2} (n+1)^2, \qquad n \in \mathbf{N}_0$$

Setting

$$\rho(x) = \frac{1}{a} \left[\cos\left(\frac{\pi x}{2a}\right) \right]^2$$

and

$$\phi_n(x) = \left\{ \begin{matrix} 0 \quad \text{for} \quad x \leq -a \\ \cos\left(\frac{n\pi x}{2a}\right) + \sin\left(\frac{n\pi x}{2a}\right) \tan\left(\frac{\pi x}{2a}\right) \\ \text{for} \quad -a < x < a \\ 0 \quad \text{for} \quad x \geq -a \end{matrix} \right\}, \quad n = 0, 2, 4, \ldots$$

$$\phi_n(x) = \begin{Bmatrix} 0 & \text{for} \quad x \leq -a \\ \sin\left(\dfrac{n\pi x}{2a}\right) + \cos\left(\dfrac{n\pi x}{2a}\right)\tan\left(\dfrac{\pi x}{2a}\right) & \\ \text{for} \quad -a < x < a & \\ 0 & \text{for} \quad x \geq -a \end{Bmatrix}, \quad n = 1, 3, 5, \ldots$$

we get the probability measure $w = \rho\mu_L$,

$$w(\mathbf{R}) = \frac{1}{a}\int_{-a}^{a} \left[\cos\left(\frac{\pi x}{2a}\right)\right]^2 dx = 1$$

which is defined on the measurable space $(\mathcal{A}_L, \mathbf{R})$.

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